

GEOMETRIC HOMOLOGY

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1. INTRODUCTION

The purpose of this paper is to introduce a version of singular homology based on smooth mappings of manifolds with corners. Although variants of such a theory exists in the literature (see [3], [1], [16]), we felt that certain points were not adequately addressed. In particular, our goal is to construct a chain level theory based on smooth mappings of manifolds with corners. In addition, we will avoid using the fact that smooth manifolds with can be triangulated. As we shall see, transversality and intersections play a major role in setting up this theory. From a pedagogical viewpoint, having intersection theory arguments available from the start facilitates simple and intuitive computations.

The motivation for our construction comes from several sources. There has been a renewed interest in defining invariants based on geometric chains mapping to configuration space. Such chains appear as solutions of various nonlinear partial differential equations in Gromov-Witten theory as well as gauge theory. In addition, geometric chains also appear in the work of Sullivan and Chas on String topology [2]. The geometric construction of various ∞ -structures is closely related to a chain level construction of intersection theory.

Ultimately, however, we are interested in the infinite dimensional variant of this theory. In [5], we introduced a new approach to Floer theory based on a notion of a semi-infinite cycle. This approach provides an alternative (but equivalent) approach to the Morse theory construction of Floer homology. The fundamental objects in this theory are semi-infinite chains which are mappings of Hilbert manifolds

$$\sigma : P \rightarrow B$$

meeting certain topological axioms. The theory we discuss in this paper is a finite dimensional analogue of this geometric construction. As it turns out, many arguments in Floer theory (for instance, the proof of the Morse Homology theorem) are nearly identical to their finite dimensional counterparts once certain topological assumptions are satisfied. For these reasons, we thought it would be useful to expose our approach to geometric homology in the finite dimensional case to help motivate the analogous arguments in Floer theory.

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2. GEOMETRIC PRELIMINARIES

2.1. Manifolds With Corners. Here we review the concept of a manifold with corners and prove various geometric lemmas that are used in the paper.

Definition 1. A map $\mathbb{R}^k \times [0, 1]^j \rightarrow \mathbb{R}^l \times [0, 1]^k$ is **smooth** if locally it is a restriction of a smooth map on $\mathbb{R}^{k+j} \rightarrow \mathbb{R}^{l+k}$.

Definition 2. Let P be a Hausdorff, 2nd countable topological space. P is a **smooth manifold with corners** if it is equipped with an open cover $\{U_i\}$ together with homeomorphisms

$$\phi_i : \mathbb{R}^k \times [0, 1]^{n-k} \rightarrow U_i$$

We demand that $\phi_j^{-1} \circ \phi_i$ are smooth where defined. The collection (U_i, ϕ_i) is an **atlas** for P . As usual, picking a maximal compatible atlas specifies a smooth structure on P .

P has a natural structure of a stratified space. Indeed, Let P^i be the set of points p with a local chart of the form $\mathbb{R}^{n-i} \times [0, 1]^i$ where $p = \{0\}$. We have

$$P = P^0 \cup P^1 \cup P^2 \cup P^3 \dots$$

One may check, using the inverse function theorem, that definition of strata is independent of the choice of chart. Note that P^i is a manifold without corners of dimension $n - i$. We will assume that, for a given P , each connected component of the top stratum has the same dimension n . Let M be a smooth manifold without boundary and $f : P \rightarrow M$ be a smooth map.

Definition 3. A map $f : P \rightarrow M$ is **transverse** to a submanifold $Y \subset M$ if its transverse on each open stratum P^i of P .

Lemma 1. Consider a manifold M with closed submanifold Y . Given a smooth map

$$\sigma : P \rightarrow M$$

such that σ is transverse to Y , $\sigma^{-1}(Y)$ is a manifold with corners.

Proof. Locally, we may write σ as

$$\sigma : \mathbb{R}^l \times [0, 1]^i \rightarrow \mathbb{R}^n$$

with $Y = \mathbb{R}^k \times 0 \subset \mathbb{R}^n$. Let $p \in \mathbb{R}^l$. We will produce a chart for $\sigma^{-1}(Y)$ near $p \times 0$. Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ be the projection to the last $n - k$ factors. The map

$$f : \mathbb{R}^l \times [0, 1]^i \rightarrow \mathbb{R}^{n-k} \times [0, 1]^i$$

given by $f(v, t) = (\pi(\sigma(v, t)), t)$ is a surjection in view of our transversality assumption. The implicit function theorem assures that we can pick coordinates $U \times [0, 1]^i$ for around p such that $\pi(\sigma(u, t)) = \phi(u)$ for some smooth function ϕ . Therefore, another application of the implicit function theorem yields that $\phi^{-1}(0) \times [0, 1]^i$ is a chart for $\sigma^{-1}(Y)$. \square

2.2. Boundary Operator. Our goal is to define a geometric boundary operator on manifolds with corners. For $[0, 1]^k$ let

$$\partial_i([0, 1]^k) = [0, 1]^{k-1}$$

with the natural face inclusion

$$s_i : \partial_i([0, 1]^k) \rightarrow [0, 1]^k$$

that omits the i th factor. For $[0, 1]^k$, let

$$\partial[0, 1]^k = \bigsqcup_i \partial_i[0, 1]^k$$

Given a P , we construct ∂P as follows. We replace the chart $\mathbb{R}^{n-k} \times [0, 1]^k$ by $\mathbb{R}^{n-k} \times \partial[0, 1]^k$. The transition functions for P induce transition functions for ∂P . Note that locally, we get one copy of P^1 , two copies of P^2 , etc.

2.3. Orientations.

Definition 4. P is said to be **orientable** if P^0 is orientable. An **orientation** of P is an orientation of P^0 .

The following definition is important for the construction of homology groups:

Definition 5. Given a manifold with corners P , a mapping $\sigma : P \rightarrow M$ is said to be **trivial**, if there exists an orientation reversing diffeomorphism $f : P \rightarrow P$ with $\sigma \circ f = \sigma$.

Lemma 2. An orientation of P induces an orientation of ∂P .

Proof. Since $(\partial P)^0 = P^1$, this is a consequence of the fact that an orientation of P^0 induces an orientation on P^1 using the “outward normal first” convention. \square

Since diffeomorphism of P induces one of ∂P , we have:

Lemma 3. If $\sigma : P \rightarrow M$ is trivial, so is $\partial\sigma : \partial P \rightarrow M$.

The following lemma will imply that $\partial^2 = 0$ in the homology theory we define:

Lemma 4. Given any σ , we have $\partial^2\sigma$ trivial.

Proof. Locally, $\partial^2(V \times [0, 1]^k) = V \times \partial^2[0, 1]^k$. Now,

$$\partial^2[0, 1]^k = \coprod_{i \neq j} \partial_i \circ \partial_j [0, 1]^k$$

Thus, $\partial^2[0, 1]^k$ is naturally a disjoint union of two spaces corresponding to when $i < j$ and $i > j$. This provides the orientation reversing involution of $\partial^2\sigma$. \square

We now recall the notion of orientable map between finite dimensional manifolds. For the general discussion, see [4]. Given a smooth map

$$\sigma : P \rightarrow M$$

of finite dimensional manifolds, recall the construction of determinant bundle

$$\det(\sigma) \rightarrow P$$

Intuitively, the line over a point $p \in P$ corresponds to the space

$$\Lambda^{max} \text{Ker} D\sigma_p \otimes \Lambda^{max} (\text{Coker} D\sigma_p)^*$$

Since in general the dimension of $\text{ker} D\sigma$ may jump, to construct a locally trivial bundle one proceeds as follows. Locally, one may pick a trivialized bundle $\underline{\mathbb{R}}^n \rightarrow P$ and a map of bundles

$$L : \underline{\mathbb{R}}^n \rightarrow \sigma^*(TM)$$

such that the map $D\sigma \oplus L : TP \oplus \underline{\mathbb{R}}^n \rightarrow \sigma^*(TM)$ is surjective. One then defines $\det(\sigma)$ as

$$\Lambda^{max}(\text{Ker}(D\sigma \oplus L)) \otimes (\Lambda^n \underline{\mathbb{R}}^n)^*$$

One can check that does not depend on the local choices and defines a line bundle over P . A useful fact in dealing with orientations is that any exact sequence:

$$0 \rightarrow V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_n \rightarrow 0$$

gives rise to a canonical isomorphism

$$\otimes_{\text{even}} \Lambda^{max} V_i \cong \otimes_{\text{odd}} \Lambda^{max} V_i$$

Definition 6. An orientation of a map σ is an orientation of $\det(\sigma)$.

We will need several basic results about oriented maps. Given transverse maps $\sigma : P \rightarrow N$ and $f : M \rightarrow N$, we may form the pullback

$$f^*(\sigma) : f^{-1}(P) \rightarrow M$$

This is defined as the subset of $M \times P$ such that $(m, p) \in f^{-1}(P)$ if $f(m) = \sigma(p)$.

Lemma 5. We have $f^*(\det(\sigma)) \cong \det(f^*(\sigma))$

Proof. The tangent space to $f^{-1}(P)$ is given by $(v, w) \in TM \times TP$ with $Df(v) = D\sigma(w)$. The kernel of the map $f^*\sigma : f^{-1}(P) \rightarrow M$ is given by points (v, w) with $v = 0$ and $D\sigma(w) = 0$. Thus, $\ker(f^{-1}\sigma)$ is naturally identified with $\ker(\sigma)$. In view of the isomorphism

$$TM/\text{im}(f^{-1}\sigma) \rightarrow TN/\text{im}(\sigma)$$

the cokernels are also naturally identified. \square

This implies that an orientation of the map σ induces one on $f^*(\sigma)$.

Lemma 6. *If f is oriented and P is oriented, so is $f^{-1}(P)$.*

Proof. The tangent space to $f^{-1}(P)$ is given by $(v, w) \in TM \oplus TP$ with $Df(v) = D\sigma(w)$. The exact sequence

$$0 \rightarrow \text{Ker} Df \rightarrow TM \rightarrow TN \rightarrow \text{Coker} Df \rightarrow 0$$

gives a trivialization of $\Lambda^{\max} TM \otimes \Lambda^{\max}(TN)^*$ while the sequence

$$0 \rightarrow Tf^{-1}(P) \rightarrow TM \oplus TP \rightarrow TN \rightarrow 0$$

implies that a trivialization of $\Lambda^{\max} Tf^{-1}(P)$ is equivalent to that of $\Lambda^{\max} TM \otimes \Lambda^{\max}(TN)^* \otimes \Lambda^{\max} TP$. \square

Note that given an oriented map $\sigma : P \rightarrow M$, we get an induced oriented map $\partial\sigma : \partial P \rightarrow M$. The following is straightforward to verify using a local chart:

Lemma 7. *Given an oriented map $\sigma : P \rightarrow M$, there exists a diffeomorphism*

$$\phi : \partial^2 P \rightarrow \partial^2 P$$

commuting with σ such that

$$\phi^*(\det(\sigma)) \cong -\det(\sigma)$$

as oriented line bundles.

2.4. Cutting by a Hypersurface. We will need to cut a manifold by a hypersurface obtaining a decomposition of the manifold into two parts. Let

$$\sigma : P \rightarrow M$$

be as above. Consider a smooth map

$$f : M \rightarrow \mathbb{R}$$

with p a regular value of $f \circ \sigma$. Let

$$\sigma^+ = \sigma|_{(f \circ \sigma)^{-1}[p, \infty)}$$

$$\sigma^- = \sigma|_{(f \circ \sigma)^{-1}(-\infty, p]}$$

and

$$\sigma^0 = \sigma|_{(f \circ \sigma)^{-1}(p)}$$

Let

$$P' = (f \circ \sigma)^{-1}(p)$$

Theorem 1. σ^\pm are manifolds with corners. Furthermore,

$$\partial(\sigma^\pm) = (\partial\sigma)^\pm \sqcup \pm\sigma^0$$

Proof. Locally we have

$$\sigma : V \times [0, 1]^k \rightarrow M$$

where V is a manifold without boundary. Applying the inverse function theorem, we can represent σ^\pm as $(\sigma|_V)^\pm \times [0, 1]^k$. The formula for the boundary follows since

$$\partial(\sigma|_V^\pm \times [0, 1]^k) = \partial\sigma|_V^\pm \times [0, 1]^k \sqcup \sigma|_V^\pm \times \partial[0, 1]^k$$

and

$$\partial\sigma|_V^\pm = \pm\sigma|_V^0$$

□

We also need to prove a result that will allow us to cut cycles into smaller pieces. We introduce a manifold $Cre(\sigma)$ that interpolates between σ and "creasing" σ along σ^0 . Let D be subset of \mathbb{R}^2 given by

$$\{(x, y) \in \mathbb{R}^2 | 0 \leq y \leq 2 - |x|, |x| < 1\}$$

D inherits the structure of a manifold with corners as a subset of the plane. We define a homeomorphism $\phi : D \rightarrow (-1, 1) \times [0, 1]$. Let

$$\phi(x, y) = (x, \frac{y}{2 - |x|})$$

Note that ϕ is a diffeomorphism outside $x = 0$.

Let $Cre(\sigma)$ be obtained as follows. As a topological space,

$$Cre(\sigma) = P \times [0, 1]$$

Now, we specify the stratification. Outside, $P' \times [0, 1]$, let the manifold structure be given by the product structure $(P - P') \times [0, 1]$. Near a point on P' , we can locally write P as $P = V \times (-\epsilon, \epsilon)$ with $f(\sigma(v, t)) = t$ and V is a manifold with corners. We take the manifold structure to be $V \times D_\epsilon$ where

$$D_\epsilon = \{(x, y) \in D | -\epsilon < x < \epsilon\}$$

Here we identify $P = V \times (-\epsilon, \epsilon) \subset V \times D_\epsilon$ as points with $y = 0$. The overlap chart map

$$V \times D_\epsilon \rightarrow V \times (-\epsilon, \epsilon) \times [0, 1]$$

is induced by restricting ϕ to

$$\phi_\epsilon : D_\epsilon \rightarrow (-\epsilon, \epsilon) \times [0, 1]$$

Since σ induces a map $Cre(\sigma)$ by composing σ with the smooth projection

$$V \times D_\epsilon \rightarrow V \times (-\epsilon, \epsilon)$$

we have

Lemma 8. σ induces smooth map on $Cre(\sigma)$. Furthermore, if σ is trivial, so is $Cre(\sigma)$.

Lemma 9. $\partial(Cre(\sigma)) = -\sigma \sqcup \sigma^+ \sqcup \sigma^- \sqcup Cre(\partial\sigma)$

Proof. We check the statement in a neighborhood of P' . We have the chart $V \times D_\epsilon$. We have

$$\partial(V \times D_\epsilon) = \partial V \times D_\epsilon \sqcup V \times \partial D_\epsilon$$

while

$$\partial D_\epsilon = (-\epsilon, \epsilon) \sqcup (-\epsilon, 0] \sqcup [0, \epsilon)$$

$\partial V \times D_\epsilon$ corresponds to $Cre(\partial P)$

$V \times (-\epsilon, \epsilon)$ corresponds to P

$V \times [0, \epsilon)$ corresponds to σ^+ and $V \times (-\epsilon, 0]$ to σ^- . □

3. DEFINITION OF THE HOMOLOGY GROUPS

Fix some countable infinite dimensional Hilbert space \mathbb{H} once and for all. To avoid set-theoretic complications, all our chains will be subsets of \mathbb{H} . Let M be a smooth paracompact manifold without boundary. We wish to construct a homology theory for M based on mappings of manifolds with corners into M . Although we focus on the case M is finite dimensional, most of the methods of this paper carry over rather directly to the case when M is an infinite dimensional paracompact Banach manifold.

Definition 7. A **chain** is a smooth map

$$\sigma : P \rightarrow M$$

where P is a compact oriented manifold with corners embedded in \mathbb{H} . Two chains $\sigma : P \rightarrow M$ and $\tau : Q \rightarrow M$ are said to be **isomorphic** if there exists an orientation preserving diffeomorphism $f : P \rightarrow Q$ such that $\tau \circ f = \sigma$.

We will assume that all the components of P have the same dimension.

Definition 8. A chain $\sigma : P \rightarrow M$ is said to be **trivial** if there exists an orientation reversing diffeomorphism $f : P \rightarrow P$ with $\sigma \circ f = \sigma$.

Definition 9. A chain $\sigma : P \rightarrow M$ is said to have **small image** if $\sigma(P) \subset g(N)$ where $g : N \rightarrow M$ is a smooth map from a manifold with corners (not necessarily compact) with $\dim(N) < \dim(P)$.

Definition 10. A chain σ is said to be **degenerate** if σ has small image and $\partial\sigma$ is isomorphic to a disjoint union of a trivial chain and a chain with small image.

As the simplest example, note that the constant map $[0, 1] \rightarrow pt$ is degenerate but the boundary does not have small image.

Lemma 10. Let τ be a trivial chain. If $\sigma \sqcup \tau$ is trivial, then σ is trivial.

Proof. We decompose σ into mutually isomorphic components as

$$\sigma = \sigma_1 \sqcup \sigma_2 \dots$$

Here, each σ_i consists of the disjoint union of isomorphic connected components of (up to orientation) of σ and no σ_i, σ_j share isomorphic components when $i \neq j$. An automorphism of σ preserved these components. Therefore, σ is trivial exactly when for each i , either the number of components of σ_i is zero when counted with orientation or each such component admits an orientation reversing isomorphism. Since τ is trivial, $\sigma \sqcup \tau$ either adds components which are trivial or adds zero components when counted with sign. Therefore, the only way $\sigma \sqcup \tau$ is trivial is if σ was already trivial. \square

Definition 11. Let $Q(M)$ be the set of chains isomorphic to $\alpha \sqcup \beta$ where α is trivial and β is degenerate. We allow α and β to be empty.

Lemma 11. If σ is in $Q(M)$, so is $\partial\sigma$.

Proof. Since the boundary of a trivial chain is trivial, we can focus on when σ is degenerate. We have $\partial\sigma = \alpha \sqcup \beta$ with α trivial and β with small image. We need to show that β is degenerate. Since $\partial^2\sigma$ is trivial, we have $\partial\alpha \sqcup \partial\beta$ trivial. By the previous lemma $\partial\beta$ is trivial, hence β is degenerate as desired. \square

The following is the key step to defining an equivalence relation based on chains in $Q(M)$:

Lemma 12. If $\sigma \sqcup \tau \in Q(M)$ for some $\tau \in Q(M)$, we have $\sigma \in Q(M)$.

Proof. We decompose σ as $\sigma_1 \sqcup \sigma_2 \dots$ as in the lemma above. The hypothesis implies that each σ_i is either trivial or has small image. Thus, we can write $\sigma = \alpha \sqcup \beta$ with α trivial and β having small image. We show that β is degenerate. We have $\partial\sigma \sqcup \partial\tau \in Q(M)$. By repeating the argument, this implies that $\partial\beta$ is a union of a small chain and a trivial chain. Therefore, β is degenerate. \square

We are ready to define our chain complex:

Definition 12. Define an equivalence on chains as follows: $\sigma \sim \tau$ if $\sigma \sqcup -\tau$ is in $Q(M)$. We denote the resulting set by $C_*(M)$.

Lemma 13. \sim is an equivalence relation. The geometric boundary operator ∂ induces the structure of a chain complex on $C_*(M)$ with addition given by disjoint union.

Proof. Reflexivity and symmetry are clear. To check transitivity, note that $\sigma \sqcup -\tau \in Q(M)$ and $\tau \sqcup -\rho \in Q(M)$ imply $\sigma \sqcup -\tau \sqcup \tau \sqcup -\rho \in Q(M)$. By the previous lemma, $\sigma \sqcup -\rho$ is in $Q(M)$ as desired. The additive structure is induced by disjoint union: Given $\sigma : P \rightarrow M$ and $\tau : Q \rightarrow M$ we must choose an embedding of $P \sqcup Q$ in our Hilbert space \mathbb{H} . However, any two such choices will be equivalent under our relation since the difference will be isomorphic to a trivial chain. $C_*(M)$ has inverses since $\sigma \sqcup -\sigma$ is trivial for any σ . Finally, $C_*(M)$ forms a complex since $\partial Q(M) \subset Q(M)$ and $\partial^2 \sigma$ is always trivial. \square

Note that $C_*(M)$ has a natural grading given by the dimension of the chains.

Definition 13. Let $H_*(M)$ denote the homology groups associated to the complex $C_*(M)$.

Let us observe that cycles with small image are 0 in $C_*(M)$:

Lemma 14. If σ has small image and $\partial \sigma \sim 0$, then $\sigma \sim 0$.

Proof. We have that $\partial \sigma \in Q(M)$. Therefore, σ is degenerate and hence also in $Q(M)$. \square

As will be proved in subsequent sections, the homology groups introduced above are isomorphic to the singular homology of M .

4. COHOMOLOGY

There is a variant of the groups that is based on proper rather than compact chains. These will give a geometric version of cohomology.

Definition 14. A **cochain** is a smooth oriented map $\sigma : P \rightarrow M$ where P is a smooth manifold with corners and σ is a proper map.

Not to confuse the two distinct notions of orientation we will say that cochains are **cooriented**. Given $\sigma : P \rightarrow M$ and $\tau : P' \rightarrow M$ and a diffeomorphism $f : P \rightarrow P'$ such that $\sigma \circ f = \tau$, we get an induced isomorphism $\det(\sigma) \rightarrow f^*(\det(\tau))$. Therefore, we can ask whether this map preserves orientations:

Definition 15. A cochain $\sigma : P \rightarrow M$ is said to be **trivial** if there exists a diffeomorphism $f : P \rightarrow P$ with $\sigma \circ f = \sigma$ which changes the orientation of the map.

The rest of the definitions go through as before and give rise to a chain complex with the boundary operator given by geometric boundary.

Definition 16. Let $C^*(M)$ be the chain complex of proper maps. We denote its homology groups by $H^*(M)$.

$C^*(M)$ is graded by the codimension of the map $\sigma : P \rightarrow M$. With this convention, ∂ has degree 1. As we will show later, $H^*(M)$ is a geometric realization of the singular cohomology of M .

Remark. In the case that M is an infinite dimensional manifold, cochains are given by proper Fredholm maps $\sigma : P \rightarrow M$, where P is necessarily infinite dimensional. The rest of the construction proceeds with little change.

5. TRANSVERSALITY AND PAIRING

In this section we use standard results on transversality to obtain a pairing:

$$\cap : H_a(M) \otimes H^b(M) \rightarrow H_{a-b}(M)$$

The arguments are quite standard, but we need pay attention to the fact that perturbations should preserve trivial and degenerate chains.

Definition 17. Maps $\sigma : P \rightarrow M$, $\tau : Q \rightarrow M$ are in **general position** if they satisfy the following conditions:

1. They are transverse on each open stratum.
2. If P_i a component of an open stratum of P with small image, then there exists a smooth map $g : T \rightarrow M$ of smaller dimension covering the image of P_i which is transverse to all the strata of τ .
3. If Q_i a component of an open stratum of Q with small image, then there exists a smooth map $g' : T' \rightarrow M$ of smaller dimension covering the image of Q_i , which is transverse to all the strata of σ .

Now we exhibit enough perturbations to ensure transversality while preserving degenerate chains. To this end, we have the following basic result:

Lemma 15. There exists a smooth connected manifold \mathfrak{P} and a smooth map

$$F : M \times \mathfrak{P} \rightarrow M$$

such that D_2F is surjective at all points. We assume $F(\cdot, p) = \text{Id}$ on M for some $p \in \mathfrak{P}$.

Using standard transversality arguments we have the following:

Lemma 16. Given $\sigma : P \rightarrow M$ and $\tau : P \rightarrow M$ there exists $a \in \mathfrak{P}$ such that $F(\cdot, a) \circ \sigma$ and τ are in general position. Given any two such $a, b \in \mathfrak{P}$, there exists a path $\gamma : [0, 1] \rightarrow \mathfrak{P}$ such that $\gamma(0) = a$, $\gamma(1) = b$ and the map

$$\Sigma : P \times [0, 1] \rightarrow M$$

given by $\Sigma(p, t) = F(\sigma(p), \gamma(t))$ is transverse to τ .

Proof. This is a standard transversality argument. We need to ensure transversality for each open stratum as well as for some choice of maps $g_i : T_i \rightarrow M$ covering components with small image. Since each chain has at most a countable number of components in each stratum, we can ensure simultaneous transversality by an application of Sard's theorem. \square

Definition 18. *Given σ and τ in general position, let $\sigma \cap \tau = (\sigma \times \tau)^{-1}(\Delta)$ where Δ is the diagonal in $M \times M$.*

Note that $\sigma \cap \tau$ is oriented and comes with a map to M . Also, since σ has compact domain and τ is proper, $\sigma \cap \tau$ is compact. The need for such specific perturbations is to ensure taking intersections preserves degenerate and trivial chains:

Lemma 17. *Assume σ and τ are in general position. If σ is in $Q(M)$, so is $\sigma \cap \tau$.*

Proof. If σ has a orientation reversing self-diffeomorphism f such that $\sigma \circ f = \sigma$, then

$$F(\cdot, a) \circ \sigma \circ f = F(\cdot, a) \circ \sigma$$

Therefore, $\sigma \cap \tau$ has an induced orientation reversing diffeomorphism as well. If σ has small image contained in $g : T \rightarrow M$, then $F(\cdot, s) \circ \sigma$ has image contained in $F(\cdot, s) \circ g$ as long as g is transverse to τ . Thus, $\sigma \cap \tau$ has small image contained in $g \cap \tau$. Furthermore, if we assume σ is degenerate, we have $\sigma \cap \tau$ has small image and

$$\partial(\sigma \cap \tau) = \partial\sigma \cap \tau \sqcup \pm \sigma \cap \partial\tau$$

is a union of s trivial chain and a chain with small image. \square

Theorem 2. *Transverse intersections induce a well defined map:*

$$\cap : H_a(M) \otimes H^b(M) \rightarrow H_{a-b}(M)$$

Proof. Given a cycle $\sigma : P \rightarrow M$ and a cocycle $\tau : Q \rightarrow M$, we will use a generic homotopy

$$F : M \times [0, 1] \rightarrow M$$

to obtain a representative $F(\cdot, 1) \circ \sigma$ of the homology class of σ that is transverse to τ . Suppose now that $\sigma \sim \partial\sigma_1$ where σ is transverse to τ . If σ_1 is also transverse to τ , then $\partial(\sigma_1 \cap \tau) \sim \sigma \cap \tau$ as desired. If not, pick a generic homotopy F as above and let

$$\sigma_2(p, t) = F(\sigma_1(p), t)$$

Since $\partial(\sigma_1 + \partial\sigma_2) \sim \sigma$, we need only check that $\sigma_1 + \partial\sigma_2$ is equivalent to a transverse chain. Let

$$\sigma_3(p, t) = F(\sigma(p), t)$$

By construction, $\sigma_1 + \partial\sigma_2 \sim \sigma_3 \sqcup \sigma_2(\cdot, 1)$. If we take a generic F , σ_3 and $\sigma_2(\cdot, 1)$ will be transverse to τ as desired. \square

By a similar technique applied to proper maps, we obtain a geometric version of the cup product:

$$H^a(M) \otimes H^b(M) \rightarrow H^{a+b}(M)$$

6. EILENBERG-STEENROD AXIOMS

In this section we verify that our theory satisfies the Eilenberg-Steenrod axioms. We begin by computing the homology of a point:

Theorem 3. $H_k(pt) = 0$ when $k \neq 0$ and $H_0(pt) \cong \mathbb{Z}$.

Proof. Since any map $\sigma : P \rightarrow pt$ is constant any such chain has small image when $\dim(P) > 0$. If σ is a cycle, we have $\partial\sigma$ a union of trivial and degenerate chains. Therefore, σ is degenerate and hence 0 in $C_*(pt)$. To verify the claim in dimension zero note that 0 dimensional cycles are finite collections of (oriented) points, while 1-dimensional cycles are compact intervals. \square

Next, we verify a version of functoriality:

Theorem 4. *Given a smooth map $f : M \rightarrow N$ between smooth manifolds, we have an induced chain map $f_* : C_*(M) \rightarrow C_*(N)$. f_* extends to a functor from the category of smooth finite dimensional manifolds and smooth maps to the category of chain complexes over \mathbb{Z} .*

Proof. It suffices to note that the image of a trivial chain is trivial (with the same orientation reversing diffeomorphism) and given a map σ with small image contained in $g : T \rightarrow M$, we have $f \circ \sigma$ has image contained in $f \circ g : T \rightarrow N$. \square

Theorem 5. *Given a smooth homotopy $F : M \times [0, 1] \rightarrow N$, we have that $F_*(\cdot, 0)$ is chain homotopic to $F_*(\cdot, 1)$.*

Proof. The chain homotopy H is defined as follows. Given a chain $\sigma : P \rightarrow M$ let $H(\sigma) : P \times [0, 1] \rightarrow M$ be the map with $H(\sigma)(p, t) = F(\sigma(p), t)$. We have $\partial H(\sigma) = (-1)^{|P|}(F(\sigma, 1) - F(\sigma, 0)) + H(\partial\sigma)$. \square

In the case f is not smooth one can still define a map on the level of homology. Indeed, one can approximate any continuous f by a smooth f' . Such a choice is unique up to smooth homotopy. Therefore, the map on homology is still well defined as follows from the proof of the homotopy axiom.

Remark. The restriction that M is smooth is not significant. Since any finite CW complex X has the homotopy type of a finite dimensional manifold (possibly noncompact), we can define our homology groups for such X by choose a homotopy equivalent smooth model for X .

Given an open set $U \subset M$ we have the exact sequence:

$$0 \rightarrow C_*(U) \rightarrow C_*(M) \rightarrow C_*(M)/C_*(U) \rightarrow 0$$

We define $C_*(M, U)$ as $C_*(M)/C_*(U)$ and obtain the long exact sequence

$$\cdots \rightarrow H_*(U) \rightarrow H_*(M) \rightarrow H_*(M, U) \rightarrow \cdots$$

Finally, we verify a version of the excision axiom. Let $U \subset M$ be an open subset and $K \subset U$ be compact set. For excision, as well as later applications, we will need to cut cycles into pieces without changing the homology class. Consider a smooth function $f : M \rightarrow \mathbb{R}$. Let

$$\begin{aligned}\sigma^+ &= \sigma|_{(f \circ \sigma)^{-1}[p, \infty)} \\ \sigma^- &= \sigma|_{(f \circ \sigma)^{-1}(-\infty, p]} \end{aligned}$$

and

$$\sigma^0 = \sigma|_{(f \circ \sigma)^{-1}(p)}$$

Lemma 18. *Given $\sigma : P \rightarrow M$ with $\sigma \in Q(M)$, we have $\sigma^\pm, \sigma^0 \in Q(M)$ for generic p . Let $\sigma : P \rightarrow M$ with $\partial\sigma \sim \tau$ where $\tau \in U \subset M$. For generic p , we have $[\sigma] = [\sigma^+ \sqcup \sigma^-]$ in $H_*(M, U)$.*

Proof. If σ is trivial so are σ^\pm and σ^0 since the involution descends to them. If σ is degenerate, we have that $\text{im}(\sigma) \subset \text{im}(g)$ for $g : T \rightarrow M$. In addition, we have that $\partial\sigma = \alpha \sqcup \beta$ with β trivial and α covered by g' of smaller dimension. Therefore, by choosing a generic p , we can assume its a regular value of g', g and σ . Now, σ^\pm has small image since σ has small image. σ^0 has image contained in g^0 and hence has small image as well. We have

$$\partial\sigma^\pm = (\partial\sigma)^\pm \sqcup \pm\sigma^0$$

while $\partial\sigma^0 = (\partial\sigma)^0$. Since α^\pm has small image and β^\pm is we see that σ^\pm and σ^0 are degenerate as well.

We apply the crease construction to conclude that

$$\partial\text{Cre}(\sigma) = \sigma^+ \sqcup \sigma^- \sqcup -\sigma \sqcup \text{Cre}(\partial\sigma)$$

Since $\partial\sigma \sim \tau$ we have $\text{Cre}(\partial\sigma) \sim \text{Cre}(\tau)$. Indeed, given a chain $\rho \in Q(M)$, we have $\text{Cre}(\rho) \in Q(M)$. This is clear for trivial chains. For degenerate chains, observe that $\text{Cre}(\rho)$ has same image as ρ and hence is small. Also,

$$\partial\text{Cre}(\rho) = \text{Cre}(\partial\rho) \sqcup -\rho \sqcup \rho^+ \sqcup \rho^-$$

This shows that $\partial\text{Cre}(\rho)$ has small image as well. Since $\text{Cre}(\tau) \in C_*(U)$, we conclude that $[\sigma] = [\sigma^+ \sqcup \sigma^-]$ in $H_*(M, U)$ as desired. \square

We can now prove the excision axiom:

Theorem 6. *The natural inclusion $i_* : H_*(M - K, U - K) \rightarrow H_*(M, U)$ is an isomorphism.*

We break the proof into two lemmas:

Lemma 19. *i_* is a surjection.*

Proof. Given a cycle σ in $H_*(M, U)$, we show that it is homologous to σ' with

$$\text{im}(\sigma') \subset M - K$$

Take a smooth function $f : M \rightarrow \mathbb{R}$ with $f = 0$ on K and $f > 1$ outside U . Let $p \in (0, 1)$ be a regular value of $f \circ \sigma$. Cutting by the preimage of p decomposes σ as $\sigma^+ + \sigma^-$ with $\text{im}(\sigma^-) \subset U$. Since, $[\sigma] = [\sigma^+ \sqcup \sigma^-]$ in $H_*(M, U)$ and $[\sigma^-] = 0$ in $C_*(M, U)$, we have $[\sigma] = [\sigma^+]$ in $H_*(M, U)$. \square

Lemma 20. *i_* is an injection.*

Proof. Consider a cycle in $H_*(M - K, U - K)$ represented by a chain σ . Suppose $[\sigma] = 0$ in $H_*(M, U)$. Thus, there exists τ and $\eta \in C_*(U)$ with $\partial\tau \sqcup -\sigma \sim \eta$. Since the image of σ avoids K , we can cut our chains by a generic hyperplane as above such that $\sigma^+ = \sigma$ and $\sigma^- = \emptyset$. Since cutting by a generic plane preserves $Q(M)$, we get

$$(\partial\tau)^+ \sqcup -\sigma \sim \eta^+$$

We have $\partial(\tau^+) = (\partial\tau)^+ \sqcup \tau^0$. Therefore,

$$\partial(\tau^+) \sqcup -\sigma \sqcup -\tau^0 \sim \eta^+$$

Since each chain in this union lies in $M - K$, $[\sigma] = 0$ in $H_*(M - K, U - K)$ as well. \square

Much of the previous discussion applies to the cohomology groups. For instance, given a smooth map $f : M \rightarrow N$ of manifolds without boundary, we get an induced map:

$$f^* : H^*(N) \rightarrow H^*(M)$$

defined by pullback. Note that it is only well defined on the homology level since we may need to perturb a cocycle representative to ensure that it is in general position. In the special case of an inclusion of an open set, $i : U \subset M$, the map i^* is well defined on the chain level since transversality is automatic. We define $H^*(M, U)$ to be the homology of the kernel of i^* .

7. THOM ISOMORPHISM

In this section we prove the Thom isomorphism theorem. As we will show, this result is a direct consequence of the definitions and does not require an inductive Mayer-Vietoris argument. Let M be a smooth connected m -manifold without boundary and $V \rightarrow M$ a real vector bundle over M of dimension n . Let $\pi : V \rightarrow M$ be the projection and $i : M \rightarrow V$ be the zero section.

Theorem 7. $\pi^* : H^*(M) \rightarrow H^*(V)$ is an isomorphism with inverse i^* . Similarly, the map $i_* : H_*(M) \rightarrow H_*(V)$ is an isomorphism with inverse π_* .

Proof. Follows immediately from the fact that both $\pi \circ i$ and $i \circ \pi$ are homotopic to identity. \square

Since we defined cohomology groups independently of homology, the fact that i_* is an isomorphism does not immediately imply the same for i^* . Let us restate the two results in geometric terms. The isomorphism for i_* simply asserts that any homology class in V may be pushed down to lie on the zero section. On the other hand, i^* identifies cocycles of different dimension. In essence, it asserts that a cohomology class is completely determined by its restriction to the zero section.

Assume that M is compact and that V is oriented. Note, that this implies that both π and i are oriented maps. Since i is proper, we get a map induced by inclusion:

$$i_* : H^*(M) \rightarrow H^{*+n}(V, V - i(M))$$

Recall that $H^{*+n}(V, V - i(M))$ is defined by considering the subcomplex of $C^*(V)$ of chains that vanish on $V - i(M)$. We have the following version of the Thom Isomorphism:

Theorem 8. $i_* : H^*(M) \cong H^{*+n}(V, V - i(M))$ with inverse induced by i^* .

Proof. The proof is very similar to the proof of excision. To show i_* is surjective consider $[\sigma] \in H^*(V, V - i(M))$. Introduce a metric on V . We may cut σ by the unit sphere bundle to get $[\sigma] = [\sigma^+ \sqcup \sigma^-]$ with σ^+ supported away from the zero section. By hypothesis, $\sigma^+ \sim 0$. Therefore, we may replace σ by a chain contained in the unit disk bundle. Projecting σ to the zero section shows that σ is homologous to a chain in $\text{im}(i_*)$. Now, we establish surjectivity. Suppose $i_*(\sigma) \sim \partial\tau$ in $C^*(V, V - i(V))$. Again, we may cut τ by the unit sphere bundle to produce τ^- in the unit disk bundle of V with $\partial\tau^- \sim \sigma$. Projecting to the zero section yields τ' with $\partial\tau' \sim \sigma$. \square

We end this section by pointing out the relationship between our proof and the more standard proofs based on the notion of a Thom class. When the bundle is oriented,

$$i : M \rightarrow V$$

may be viewed as a cohomology class $[U] \in H^n(V, V - i(V))$. The restriction of $[U]$ is any fiber V_p is the origin and hence the generator of $H^n(V_p, V_p - V_p^*)$. This is the characterization of the Thom class. The isomorphism

$$i_* : H^*(M) \rightarrow H^{*+n}(V, V - i(M))$$

may be viewed as the composition of pulling back by

$$\pi^* : H^*(M) \rightarrow H^*(V)$$

followed by intersecting (the geometric analogue of the cup product) with the class $[U]$. Note that the pullback, $i^*([U]) \in H^n(M)$ is the Euler class associated to the bundle V . This corresponds the intersection of the zero section with a generic section.

One can also recast the Thom isomorphism in a slightly different form. Define $H_c^*(M)$ to be the group generated by oriented cochains which have compact domain. If V is oriented, we have

$$i_* : H^*(M) \cong H_c^{*+n}(V)$$

induced by inclusion.

8. MORSE HOMOLOGY THEOREM

In this section we prove the Morse homology theorem identifying the homology groups of the complex we constructed with those of the Morse complex. There are many versions of this theorem in the literature. We decided to include a proof since it has a direct generalization to the infinite dimensional theory.

8.1. Passing a Critical Point. Let g be a Riemannian metric and

$$f : M \rightarrow \mathbb{R}$$

be a proper Morse function, i.e. all critical points of f are nondegenerate. We assume that f is bounded below. Choose a metric on M such that ∇f is complete. We also assume that there is at most one critical point at each level, although this is not essential for the arguments that follow. Given a flow line $\gamma : [a, b] \rightarrow M$ with $\gamma'(t) = -\nabla f$ we define the energy of γ as

$$\int_a^b |\gamma'(t)|^2 dt = f(a) - f(b)$$

Let $M^c = f^{-1}((-\infty, c))$. We let $\mathcal{F}_t(M)$ be the time t downward gradient flow. Thus,

$$\mathcal{F}_t \circ \mathcal{F}_\tau = \mathcal{F}_{t+\tau}$$

The flow acts on chains by $\mathcal{F}_t(\sigma) = \mathcal{F}_t \circ \sigma$.

First we show that only the presence of critical points can change the topology:

Lemma 21. *Given $a < b$. Assume there are no critical values of f in $[a, b]$. Let $i : M^a \rightarrow M^b$ be the inclusion. We have the isomorphism*

$$i_* : H_*(M^a) \rightarrow H_*(M^b)$$

Proof. Given a cycle $\sigma \in H_*(M^b)$, $\mathcal{F}_t(\sigma)$ lies in M^a for sufficiently large t . Since $\mathcal{F}_t(\sigma)$ is homologous to σ , we have that i_* is surjective. Now, assume $i_*(\sigma) = 0$. Thus, $\partial\tau \sim \sigma$ with $\tau \in H_*(M^b)$. For large t , we have $\mathcal{F}_t(\tau) \in C_*(M^a)$ and $\partial\mathcal{F}_t(\tau) \sim \mathcal{F}_t(\sigma)$. Since the gradient flow preserves the homology class, we have $[\sigma] = 0$ in $H_*(M^a)$ as well. \square

An easy compactness argument shows that:

Lemma 22. *Assume the f has a unique isolated critical point y at level c . Choose $c' > c$ so close that $M^{c'}$ has no new critical points. Let U be a small neighborhood of y . There exists $\epsilon > 0$ such that any chain $\sigma : P \rightarrow M^{c'}$ may be pushed by the flow down to σ_0 with $\text{im}(\sigma_0) \subset U \cup M^{c-\epsilon}$.*

Let V be a finite dimensional vector space with $\dim(V) = \dim(M)$. The Morse lemma allows us to identify a neighborhood U of y with a neighborhood of the origin in $V = V^+ \oplus V^-$ and

$$f : V \rightarrow \mathbb{R}$$

with

$$f(v) = -|v^-|^2 + |v^+|^2 + f(y)$$

Let

$$\text{ind}(y) = \dim(V^-)$$

By taking ϵ to be sufficiently small in the previous lemma, we can assume that U contains the $2\sqrt{\epsilon}$ ball in V .

Theorem 9. *Let $k = \text{ind}(y)$. For $\epsilon > 0$ small, $H_k(M^{c+\epsilon}, M^{c-\epsilon}) = \mathbb{Z}$ and $H_j(M^{c+\epsilon}, M^{c-\epsilon}) = 0$ for $j \neq k$.*

Proof. Let $D_{2\sqrt{\epsilon}}$ denote the closed disk centered at 0 of radius $2\sqrt{\epsilon}$ in V and let $D^\pm(2\sqrt{\epsilon})$ denote the closed disk in V^\pm . By flowing σ using the downward gradient flow, we may assume that the image of σ is contained in $M^{c-\epsilon} \cup D_{\sqrt{\epsilon}}$. By a small perturbation, we may also assume that σ is transverse to $D^+(2\sqrt{\epsilon})$.

Our next task is to flatten out σ . Pick a smooth function

$$\phi : V \rightarrow [0, 1]$$

Let ϕ be 1 on $D_{\sqrt{\epsilon}}$ and 0 outside $D_{2\sqrt{\epsilon}}$. Consider the map

$$H_t : V \rightarrow V$$

with

$$H_t(v^-, v^+) = (v^-, (1 - t\phi(v))v^+)$$

and $t \in [0, 1]$. $H_0 = \text{Id}$ while H_1 maps V^+ to 0 on $D_{\sqrt{\epsilon}}$. $H_t \circ \sigma$ provides the desired homotopy that flattens out σ near the origin. Let $\sigma_0 = H_1 \circ \sigma$. Note that

$$f(\sigma_0(p)) \leq f(\sigma(p))$$

Thus, we can assume that points in P that don't land in $M^{c-\epsilon}$ map to V^- in the local coordinates. Also note that 0 is a regular value of this map since σ was transverse to $D^+(2\sqrt{\epsilon})$.

We may cut our cycle σ_0 into two pieces $\sigma_0^- \sqcup \sigma_0^+$ by taking the preimage of a sufficiently small sphere $S^-(\delta)$ of radius $\delta < \sqrt{\epsilon}$ around the origin in V^- . Here, we take σ_0^+ to be the piece contained inside the ball $D^-(\delta)$ bounded by the sphere. By using

the crease construction, we have $[\sigma] = [\sigma_0^- \sqcup \sigma_0^+]$ in $H_j(M^{c+\epsilon}, M^{c-\epsilon})$. Now argument divides into cases depending on the dimension of the cycle.

Case $j < k$: We have that σ_0^+ is empty and thus $\sigma_0 = \sigma_0^-$ is contained in the region where $f < c$. It may be pushed to lie in $M^{c-\epsilon}$ by the flow. Therefore, $H_j(M^{c+\epsilon}, M^{c-\epsilon}) = 0$.

Case $j = k$: Here σ_0^+ is a finite collection of disks mapping diffeomorphically to $D^-(\delta)$. Note that for $j = k$, $D^-(2\sqrt{\epsilon})$ defines an element in $H_k(M^{c+\epsilon}, M^{c-\epsilon})$. By replacing σ with $\sigma \sqcup m[D_\epsilon^-]$, we can assume that $\sigma_0^+ \sim 0$ in $C_*(M^{c+\epsilon})$. Therefore, $[\sigma] + m[D_\epsilon^-]$ is equivalent to a cycle strictly below c and hence 0 in homology. Thus, $D^-(2\sqrt{\epsilon})$ generates the homology in this dimension. It's a free generator in $H_k(M^{c+\epsilon}, M^{c-\epsilon})$ since its intersection with $D^+(\sqrt{2\epsilon})$ is 1.

Case $j > k$: In this final case, we note that since σ_0^+ maps to $D^-(\delta)$ it has small image as it is contained in a manifold of smaller dimension. We have

$$\partial\sigma_0^+ = (\partial\sigma_0)^+ \sqcup \sigma_0^0$$

Since $\partial\sigma_0 \sim 0$ we have $(\partial\sigma_0)^+ \sim 0$. Also, σ_0^0 maps to $S^-(\delta)$ and hence has small image. Therefore, σ_0^+ is degenerate. Thus, σ is homologous to a cycle with image below the critical point and is therefore nullhomologous in $H_j(M^{c+\epsilon}, M^{c-\epsilon})$. \square

More generally, we may assume that a critical level has k critical points all of the same index. In that case,

$$H_*(M^{c+\epsilon}, M^{c-\epsilon}) \cong \mathbb{Z}^k$$

This proof is along the same lines as the one above.

8.2. Self-Indexing Case. In the previous section, we saw that if there is a unique nondegenerate critical point y at $f(y) = c$ then $H_j(M^{c+\epsilon}, M^{c-\epsilon}) = \mathbb{Z}$ when $j = \text{ind}(y)$ and 0 else. Suppose that the next critical point z above y is at $f(z) = d$ with

$$\text{ind}(z) = \text{ind}(y) + 1$$

Consider the composite map:

$$H_{j+1}(M^{d+\epsilon}, M^{d-\epsilon}) \xrightarrow{\partial_*} H_j(M^{d-\epsilon}) \xrightarrow{i_*} H_j(M^{d-\epsilon}, M^{c-\epsilon})$$

Since we assumed there are no other critical points between y and z , we have:

$$H_*(M^{d-\epsilon}, M^{c-\epsilon}) \cong H_*(M^{c+\epsilon}, M^{c-\epsilon}) \cong \mathbb{Z}$$

We wish to compute $i_* \circ \partial_*$. Since $H(M^{d+\epsilon}, M^{d-\epsilon})$ is generated by the unstable disk $[D_z^-(\epsilon)]$,

$$i_* \circ \partial_*([D_z^-(\epsilon)]) = [\partial D_z^-(\epsilon)]$$

viewed as an element of $H_*(M^{d-\epsilon}, M^{c-\epsilon})$. Using the gradient flow, we may push this cycle down to $M^{c+\epsilon}$. As discussed above, the image in $H_*(M^{c+\epsilon}, M^{c-\epsilon})$ is computed by counting intersections of the cycle with the disk $D_y^+(\epsilon)$ in a neighborhood of d . This is a version of the familiar statement that boundary operator may be computed by counting the intersections of attaching/belt spheres.

Now, assume that M is compact and f is self indexing. Thus, the critical points of index i are on level i of f . We have a filtration

$$\emptyset = M^{-1+\epsilon} \subset M^{0+\epsilon} \subset \dots M^{n-1+\epsilon} \subset M^{n+\epsilon} = M$$

with $H(M^{j+\epsilon}, M^{j-1+\epsilon}) = \mathbb{Z}^k$, where k is the number of critical points of index j .

Definition 19. Let $C_*^f(M) = (\oplus_j C_j^f, \partial^f)$ be the free chain complex with

$$C_j^f = H_*(M^{j+\epsilon}, M^{j-1+\epsilon})$$

The differential $\partial^f : C_*^f \rightarrow C_{*-1}^f$, $\partial^f = i_* \circ \partial_*$ arises from composing

$$\partial_* : H_j(M^{j+\epsilon}, M^{j-1+\epsilon}) \rightarrow H_{j-1}(M^{k+1})$$

with

$$i_* : H_{j-1}(M^{j-1+\epsilon}) \rightarrow H_{j-1}(M^{j-1+\epsilon}, M^{j-2+\epsilon})$$

Let $H_*^f(M)$ be the homology of the chain complex $C_*^f(M)$.

Theorem 10. Given a self-indexing Morse function f as above, we have $H_*^f(M) \cong H_*(M)$

Proof. Given the local computation, the remaining algebraic argument is identical to the proof of the cellular homology theorem (see for example [10]). \square

Let us summarize what we have accomplished. Our only assumption on f is that it is Morse and not necessarily Morse-Smale. From this we deduced the existence of a chain complex generated by the critical points whose homology is isomorphic to the geometric singular homology of M . The arguments, as well as those in the next section, make no use of gluing theory for gradient flow trajectories that is crucial in the Floer construction of the Morse complex. Furthermore, when applied to a suitable infinite dimensional variant of the theory, they give an alternative construction of the Floer chain complex.

8.3. General Case. The results of the previous section are not sufficient for applications in Floer theory. The main issue is that the assumption that the Morse function is self-indexing is too restrictive. Indeed, the perturbations in Floer theory are usually taken to be small and it is not clear to us how to perturb the relevant functional to obtain a self-indexing one. In this section we explain how given that ∇f is Morse-Smale, we can define the Morse chain complex and show that its homology coincides with the geometric cycle homology.

We will make use of the following basic compactness result (see [13]):

Theorem 11. *Let x and y be critical points of f .*

Given a sequence of possibly broken trajectories

$$\gamma_i : [a_i, b_i] \rightarrow M$$

with $\gamma_i(a_i) \rightarrow x$ and $\gamma_i(b_i) \rightarrow y$, some subsequence converges to a possibly broken trajectory between x and y .

Assume we are given a proper Morse function f which is Morse-Smale with respect to some metric on M . The Morse-Smale condition implies that given critical points x and y with $\text{ind}(x) \leq \text{ind}(y)$, there are no gradient flow lines from x to y . In fact, there exist open neighborhoods $x \in U_x$ and $y \in U_y$ such that there are no gradient flow lines from U_x to U_y . Indeed, arguing by contradiction we can shrink the sets U_x , U_y to obtain in the limit a possibly broken gradient flow line from x to y .

Definition 20. *A chain σ is said to be k -small if each critical point of index $j \geq k$ has a neighborhood U such that $U \cap \mathcal{F}_t(\sigma) = \emptyset$ for all $t \geq 0$.*

Definition 21. *Let $C_*^k \subset C_*(M)$ be the subcomplex generated by k -small chains. We have the filtration:*

$$C_*^0 \subset C_*^1 \subset \dots \subset C_*^{\dim(M)+1} = C_*(M)$$

Lemma 23. *$H_j(C_*^k) = 0$ when $j > k$.*

Proof. Given a cycle

$$\sigma : P \rightarrow M$$

in C_j^k let us see what happens as we try to push it down by the flow. As we attempt to push it past a critical point x , two things can occur. If $\text{ind}(x) > k$, $\mathcal{F}_t(\sigma)$ stays away from an entire neighborhood of the critical point. Therefore, we focus on flowing σ past a critical point with $\text{ind}(x) \leq k < j$. We will perturb σ to σ' in a small neighborhood of x to be transverse to the stable manifold of x . Here, we use the local stable manifold V^+ of the local model V^+ provided by the Morse lemma. A crucial point is that this perturbation does not take us out of C_*^k . Indeed, since the flow is Morse-Smale, a small neighborhood of x contains no points flowing arbitrary close to the critical points of index $\geq k$. Therefore the homotopy

$$\Sigma : P \times [0, 1] \rightarrow M$$

perturbing σ has image different from σ only in a small neighborhood of x . We have $j > k$, therefore σ' intersects the stable manifold in a manifold of dimension at least 1. By the arguments in the section on passing a critical points, we can modify σ' to σ'' locally around x so that σ'' lies below x and $[\sigma''] = [\sigma]$ in $H_*(C^k)$. By repeating this a finite number of steps, we can modify σ to be empty and thus 0 in $H_j(C_*^k)$. \square

Lemma 24. $H_j(C_*^k, C_*^{k-1}) = 0$ for $j \neq k$.

Proof. The argument is quite similar to the proof of the previous lemma. We are given a chain $\sigma \in C_j^k$ and $\partial\sigma \in C_{j-1}^{k-1}$. As before, we are trying to push our chain past a critical point x . We need to check that all the constructions used in the lemma regarding passing a critical point carry over to this setting. First we need perturb the chain in a neighborhood of x to be transverse to the stable manifold. The perturbation gives a chain

$$\Sigma : P \times [0, 1] \rightarrow M$$

with $\Sigma|_{P \times 1} = \sigma$ and

$$\partial\Sigma = \Sigma|_{P \times 1} \sqcup \Sigma|_{P \times 0} \sqcup \Sigma|_{\partial(P) \times [0, 1]}$$

Note that $\Sigma|_{\partial(P) \times [0, 1]} \in C_*^{k+1}$ since the perturbation is supported in a neighborhood of x and thus is zero in C_*^k/C_*^{k+1} . Therefore, we can perturb the cycle to be transverse without changing its homology class. Next, we project the cycle to manifold V^- . This does not change our class. If $j < \text{ind}(x)$ the intersection with V^+ is empty hence we can flow past it. Otherwise we cut our σ to produce $\sigma^+ \sqcup \sigma^-$ with σ^- strictly below the critical point. Since $j > \text{ind}(x)$, σ^+ is degenerate and thus can be discarded. Therefore we can flow past any critical point x without altering the homology class. Eventually, the class is empty. \square

Take $D^\pm(x_m)$ to be a disk in $V^\pm(x_m)$ around each critical point x_m of index k small enough to be in U_{x_m} . Note that $D^-(x_m)$ is a cycle in C_k^k/C_k^{k-1} . Indeed, the boundary of such a disk can flow only to a critical point of index $\leq k-1$ and thus contained in C_{k-1}^k . Our next task is to verify that these disks freely generate $H_k(C_*^k, C_*^{k-1}) \cong \mathbb{Z}^n$. We say that chain σ lies below $S \subset M$ if $\mathcal{F}_t(\sigma)$ is disjoint from S for all $t \geq 0$.

Lemma 25. Any chain in C_*^k can be pushed down by the flow to lie below $\cup_m \partial D^+(x_m)$

Proof. Consider $\mathcal{F}_t(\sigma)$ for t very large. We claim that

$$\text{im}(\mathcal{F}_t(\sigma)) \cap (\cup_m \partial D^+(x_m)) = \emptyset$$

for all t sufficiently large. By contradiction, assume there are large times t_i and trajectories starting at points of σ and ending on some ∂D_m . By compactness such trajectories must approach some critical point arbitrarily closely somewhere along the way. Let y be such a critical point. Note that $\text{ind}(y) \leq k$ since $\sigma \in C_*^k(M)$. We claim that $\text{ind}(y) < k$. First, if $y = x_m$, the value of f along the trajectory would lie strictly below ∂D_m^+ . If $y \neq x_m$, part of the trajectory represents a flow line between small neighborhoods of critical points of the same index. This is not possible since $D^+(x_m)$ is chosen to lie in U_{x_m} . \square

Lemma 26. $H_k(C_*^k, C_*^{k-1}) \cong \mathbb{Z}^n$, where n is the number of critical points of index k .

Proof. We define a map

$$Ev : H_k(C_*^k, C_*^{k+1}) \rightarrow \mathbb{Z}^n$$

which will turn out to be an isomorphism. By the previous lemma, we can push any $\sigma \in H_k(C_*^k, C_*^{k+1})$ to lie below $\cup_m \partial D^+(x_m)$. We may perturb such σ to be transverse to each $D^+(x_m)$ by altering it near x_m . Let

$$Ev(\sigma) = (a_{x_1}, a_{x_2}, \dots)$$

be the intersections of σ with $D(x_i)$. Note that this does not depend on the small perturbation since $\partial\sigma$ lies in C_*^{k-1} and thus avoids U_{x_m} . We now assert that this gives rise to a well defined intersection with the $D^+(x_m)$. To see this, take σ below $\cup_m \partial D^+(x_m)$ and assume $\sigma = \partial\tau$ with $\tau \in C_*^k$. Taking T sufficiently large $\mathcal{F}_T(\tau)$ lies below $\cup_m \partial D^+(x_m)$ and $\partial\mathcal{F}_T(\tau) = \mathcal{F}_T(\sigma)$. However, σ and $\mathcal{F}_T(\sigma)$ are homotopic on the complement of $\cup_m \partial D^+(x_m)$. Therefore, they have the same intersection with $\cup_m D^+(x_m)$. Thus, Ev vanishes on boundaries and is therefore well defined. Ev is surjective since $D^-(x_a) \cap D^+(x_b) = \delta_{ab}$. Ev is also injective since if σ has zero algebraic intersection with all $D^+(x_m)$ we can modify it to lie below all critical points of index k , as in the previous lemmas. \square

With these results in place, we can proceed to define the Morse complex as

$$C_i^f(M) = H_i(C_*^i, C_*^{i-1})$$

The differential $\partial^f : C_*^f \rightarrow C_{*-1}^f$, $\partial^f = \partial_* \circ i_*$ arises from the connecting homomorphism:

$$\partial_* : H_k(C_*^k, C_*^{k-1}) \rightarrow H_{k-1}(C_*^{k-1})$$

and the one induced by inclusion:

$$i_* : H_{k-1}(C_*^{k-1}) \rightarrow H_{k-1}(C_*^{k-1}, C_*^{k-2})$$

Geometrically, the differential may be interpreted as follows. The generators for $H_k(C_*^k, C_*^{k-1})$ can be taken to be small oriented disks $D_{x_m}^-$ in the unstable manifolds around the critical points as above. Let $D_{y_l}^+$ be the stable cooriented disks around critical points of index $k-1$. Take the intersections $\mathcal{F}_T(\partial D^-(x_m)) \cap D^+(y_l)$ for T large. These give you the coefficients of the incidence matrix. By the same arguments as in [10], we have the following Morse Homology Theorem:

Theorem 12. *There exists an isomorphism*

$$H_*(M) \cong H_*^f(M)$$

Finally, we would like to remark that there is a version of the Morse homology theorem for cohomology. In this case, one considers the upward gradient flow which exists as long as ∇f is complete. We can define $H_f^*(M)$ much like before. We have:

Theorem 13. *There exists an isomorphism*

$$H^*(M) \cong H_f^*(M)$$

9. EQUIVARIANT THEORY

9.1. Basic Construction. The theory of geometric cycles has an equivariant generalization. Below, we describe a construction which can be viewed as the geometric analogue of the Cartan construction for deRham theory. A similar theory for singular homology appears in [14]. For this section, assume M has a smooth action

$$S^1 \times M \rightarrow M$$

Definition 22. Let J be the map taking $\sigma : P \rightarrow M$ to

$$J(\sigma) : S^1 \times P \rightarrow M$$

with

$$J(\sigma)(e^{i\theta}, p) = e^{i\theta} \sigma(p)$$

We define our complex as follows. Let

$$C_*^+ = (C_*(M) \otimes \mathbb{Z}[u], \partial_J)$$

with u a formal variable of degree -2 and

$$\partial_J(\sigma u^k) = (\partial\sigma)u^k + J(\sigma)u^{k+1}$$

The grading of σu^k is $\dim(\sigma) - 2k$.

Lemma 27. $\partial_J^2 = 0$.

Proof. We compute

$$\partial_J^2 \sigma = \partial_J(\partial(\sigma) + J(\sigma)u) = \partial^2(\sigma) + J(\partial\sigma)u + \partial(J\sigma)u + J^2(\sigma)u^2$$

Observe that since $J(P)$ is already S^1 invariant

$$\text{im } J^2 \sigma = \text{im}(J\sigma)$$

and

$$\text{im}(\partial J^2 \sigma) = J(\partial J(\sigma))$$

By definition, this implies that $J^2(P)$ is degenerate and hence zero in $C_*(M)$.

$$J(\partial\sigma) : S^1 \times \partial P \rightarrow M$$

while $\partial(J\sigma) : \partial(S^1 \times P) \rightarrow M$. Therefore, $J(\partial\sigma) = -\partial(J\sigma)$. □

Definition 23. Let $H_*^+(M)$ be the homology associated with the complex $C_*^+(M)$.

We have the following analogue of the Gysin sequence:

Theorem 14. *There is a long exact sequence:*

$$\cdots \rightarrow H_{k+2}^+(M) \rightarrow H_k^+(M) \rightarrow H_k(M) \rightarrow \cdots$$

Proof. We have an exact sequence of complexes:

$$0 \rightarrow C_{*+2}^+(M) \xrightarrow{u} C_*^+(M) \rightarrow C_*(M) \rightarrow 0$$

□

Theorem 15. *Given a smooth S^1 -equivariant map $f : M \rightarrow N$, we have an induced map*

$$f_* : H_*^+(M) \rightarrow H_*^+(N)$$

Proof. We define $f_*(\sigma u^k)$ as $(f \circ \sigma)u^k$. If $\sigma : P \rightarrow M$, we have

$$J(f \circ \sigma) : S^1 \times P \rightarrow N$$

with

$$J(f \circ \sigma)(e^{i\theta}, p) = e^{i\theta} f(\sigma(p))$$

Equivariance of f implies that

$$f_*(J(\sigma))(e^{i\theta}, p) = f(e^{i\theta} \sigma(p)) = e^{i\theta} f(\sigma(p))$$

Therefore, $J(f_*) = f_*(J)$ as desired. □

The homotopy axiom is also easy to verify:

Theorem 16. *Given an equivariant map $H : M \times [0, 1] \rightarrow N$, we have that $H_*(\cdot, 0) = H_*(\cdot, 1)$.*

We now introduce two variants of the equivariant construction:

Definition 24. *Let $H_*^\infty(M)$ be the homology associated with the complex*

$$C_*^\infty(M) = (C_*(M) \otimes \mathbb{Z}[u, u^{-1}], \partial_J)$$

and $H_^-(M)$ be the homology of the complex*

$$C_*^-(M) = C_*(M) \otimes \mathbb{Z}[u, u^{-1}] / u \cdot (C_*(M) \otimes \mathbb{Z}[u])$$

Since localization is exact, we have:

Theorem 17. $H_*^\infty(M) \cong u^{-1} H_*^+(M)$

Theorem 18. *There is a long exact sequence:*

$$\cdots \rightarrow H_{k+2}^+(M) \rightarrow H_k^\infty(M) \rightarrow H_k^-(M) \rightarrow \cdots$$

Proof. This follows from the short exact sequence of complexes:

$$0 \rightarrow C_{*+2}^+(M) \xrightarrow{u} C_*^\infty(M) \rightarrow C_*^-(M) \rightarrow 0$$

□

Remark. One can show (see [15] for a discussion) that $H_*^\infty(M)$ is the contribution to the homology coming from the fixed points of the action. In particular, one has the localization formula:

$$H_*^\infty(M) \cong H_*(M^{\text{fix}}) \otimes \mathbb{Z}[u, u^{-1}]$$

In the case M is infinite dimensional, one must consider a completion of the groups with respect to u for the localization formula to hold.

9.2. Cohomology and Pairing. We can define cohomological versions of these groups that will turn out to be dual to the ones in the previous section.

Definition 25. Let v be a formal variable of degree 2. Let $H_+^*(M)$ (resp. $H_-^*(M)$, $H_\infty^*(M)$) be the group associated to the complex $(C^*(M) \otimes \mathbb{Z}[v], \partial_J)$ (resp. $(C^*(M) \otimes \mathbb{Z}[v, v^{-1}], \partial_J)$, $(C^*(M) \otimes \mathbb{Z}[v, v^{-1}]/v(C^*(M) \otimes \mathbb{Z}[v]), \partial_J)$).

Now, we discuss the cohomology pullback map which will use transversality theory. We state the results for the “+” version, but they have straightforward generalizations to all the versions.

Theorem 19. Given an equivariant map $f : M \rightarrow N$, we have an induced map $f^* : H_+^*(N) \rightarrow H_+^*(M)$. This construction is functorial.

Proof. Given a cocycle $\sigma_0 + \sigma_1 v \cdots$ our first task is to perturb it to be transverse to f . We must not change the equivariant cohomology class during the perturbation. Using usual transversality arguments, let $\partial\tau_0 = \sigma_0 - \sigma'_0$ with σ'_0 transverse to f . Therefore, we may alter our class to be

$$\sigma'_0 + (\sigma_1 - J\tau_0)v + \cdots$$

Now, perturb $\sigma_1 - J\tau_0$ to be transverse to f . This time the correction lies in the v^2 term. We may continue in this fashion until the terms become degenerate. Therefore, we may assume that $\sigma = \sigma_0 + \sigma_1 v \cdots$ is transverse to f . We set

$$f^*(\sigma) = \sigma_0 \cap f + \sigma_1 \cap f v \cdots$$

We need to check that f_* commutes with ∂_J on transverse chains. It is clear that

$$\partial(f^*(\sigma)) = f^*(\partial\sigma)$$

If

$$\sigma_i : P \rightarrow N$$

we have

$$f^*(\sigma_i) : Q \rightarrow M$$

where $Q = (p, m)$ with $f(m) = \sigma(p)$. Therefore,

$$J(f^*(\sigma_i)) : S^1 \times Q \rightarrow M$$

Meanwhile,

$$f^*(J\sigma_i) : Q' \rightarrow M$$

where $Q' = (e^{i\theta}, p, m)$ with $f(m) = e^{i\theta} \sigma_i(p)$. The diffeomorphism

$$\phi : Q \rightarrow Q'$$

sending $(e^{i\theta}, p, m)$ to $(e^{i\theta}, p, e^{i\theta} m)$ provides the isomorphism of $J(f^*(\sigma_i))$ and $f^*(J(\sigma_i))$. Therefore, $\partial_J(f^*(\sigma)) = f^*(\partial_J \sigma)$ as desired. \square

By a similar pullback argument, we have the homotopy axiom:

Theorem 20. *Given an equivariant map $H : M \times [0, 1] \rightarrow N$, we have that $H^*(\cdot, 0) = H^*(\cdot, 1)$*

While transversality arguments may fail in the equivariant context, the previous results give a way of providing using proofs based on general position by keeping track of the "error" terms. This idea will be significant in the construction of Chern classes below.

One may use transversality to pair homology with cohomology:

Lemma 28. *There is a pairing $H_-^a(M) \otimes H_a^+(M) \rightarrow \mathbb{Z}$*

Proof. Let us say that σ is **strongly transverse** to τ $\tau \pitchfork \sigma$ and $\tau \pitchfork J(\sigma)$. Given chains $\Sigma_{i=0}^n \sigma_i v^i \in C_*^+(M)$ and $\Sigma_{i=-n}^0 \tau_i u^i \in C_*^-(M)$ with σ_i strongly transverse to τ_{-i} we define the intersection as

$$\Sigma_{i=0}^n \#(\sigma_i \cap \tau_{-i}) \in \mathbb{Z}$$

The calculation below is based on the following observation. Given $\sigma : P \rightarrow M$ and $\tau : Q \rightarrow M$ where σ has dimension i and τ has codimension $i - 1$, we have

$$\#(J(\sigma) \cap \tau) = \#((-1)^{|\sigma|+1} \sigma \cap J(\tau))$$

Indeed, we have the bijection

$$(e^{i\theta}, p, q) = (p, e^{-i\theta}, q)$$

for triples with $e^{i\theta} \sigma(p) = \tau(q)$.

We have

$$\begin{aligned} & \#(\partial_J(\sigma) \cap \tau + (-1)^{|\sigma|} \sigma \cap \partial_J \tau) \\ &= \Sigma_{i=0}^n \#((\partial \sigma_i + J \sigma_{i-1}) \cap \tau_{-i} + (-1)^{|\sigma|} \sigma_i \cap (\partial \tau_{-i} + J \tau_{-i-1})) \\ &= \Sigma_{i=0}^n \#((\partial \sigma_i) \cap \tau_{-i}) + (-1)^{|\sigma|} \#(\sigma_i \cap \partial \tau_{-i}) = \Sigma_{i=0}^n \#(\partial(\sigma_i \cap \tau_{-i})) = 0 \end{aligned}$$

This calculation implies that the intersection count is well defined. To complete the construction we must check that the chains may be perturbed to be transverse without changing the homology class. Assume inductively that σ_0 through σ_{k-1} are strongly transverse to all τ_j . Let

$$\rho : P_k \times [0, 1] \rightarrow M$$

be a generic homotopy of σ_k such that $\rho(\cdot, 0) = \sigma_k$ and $\rho(\cdot, 1) = \sigma'_k$ is strongly transverse to τ_{-j} . Taking $\partial_J \rho$ we may replace σ_k by $\sigma'_k + \rho'$ where ρ' is a generic homotopy of $\partial \sigma_k$. However, $\partial \sigma_k \sim J(\sigma_{k-1})$ since σ is a cycle. Since σ_{k-1} is strongly transverse to all τ_j , we may assume by a generic choice of ρ that ρ' is equivalent on the level of chains to a strongly transverse representative. After finitely many steps, we obtain a chain where the only non transverse chains have coefficient u^l so large that τ_{-l} is empty. \square

Remark. It is not immediate how to define a cup product on $H_+^*(M)$ directly. Indeed, on $M \times M$ with the diagonal S^1 -action, the operator J does not decompose as an operator on the factors. Thus, given cohomology classes $a, b \in H_+^*(M)$, $a \times b$ does not define an equivariant class in $M \times M$. In fact, the difference

$$\{a, b\} := J(a \cap b) - J(a) \cap b - (-1)^{|a|} a \cap J(b)$$

is a finite dimensional analogue of the string bracket of Chas-Sullivan (see [2]).

9.3. Some Calculations. Here are a few simple calculations:

Lemma 29. $H_*^+(pt) \cong \mathbb{Z}[u]$; $H_*^\infty(pt) \cong \mathbb{Z}[u, u^{-1}]$; $H_*^-(pt) \cong \mathbb{Z}[u, u^{-1}]/\mathbb{Z}[u]$

Proof. These follow directly from the fact that for a point, both J and ∂ are zero. \square

Lemma 30. $H_*^+(S^1) \cong \mathbb{Z}$; $H_*^\infty(S^1) \cong 0$; $H_*^-(S^1) \cong \mathbb{Z}$

Proof. $H^+(S^1)$: Let $\sigma = \sigma_0 + \sigma_1 u + \sigma_2 u^2 \cdots$ be a cycle and let

$$id : S^1 \rightarrow S^1$$

be the identity map.

Suppose σ is a cycle of degree > 1 . Therefore, σ_i is automatically small. We have $\partial \sigma_0 \sim 0$, and thus σ_0 is degenerate while σ_i is degenerate for $i > 0$ simply because $\dim(\sigma_i) > 2$.

Suppose σ has degree 1. Since $\dim(\sigma_i) > 2$ for $i > 0$, $\sigma_i \sim 0$ for $i > 0$. Furthermore, $\partial_J \tau = \partial \tau_0$. Therefore, in degree 1, we are reduced to the ordinary homology of S^1 which has id as a free generator.

Suppose σ has degree 0. $\sigma_i \sim 0$ for $i > 1$ since all such chains have $\dim > 2$ and are therefore degenerate. Thus, $\sigma = \sigma_0 + \sigma_1 u$ with $J(\sigma_0) \sim -\partial \sigma_1$. σ_0 is a finite collection of oriented points. Therefore, $J\sigma_0$ is a finite number of copies of i . By the computation of the ordinary homology of S^1 , $J(\sigma_0) \sim -\partial \sigma_1$ implies that the algebraic count of σ_0 is trivial. Thus, there exists τ_0 with $\partial \tau_0 = \sigma_0$. Therefore, we can assume that $\sigma = \sigma_1 u$. In this case, σ_1 is degenerate since it has small image and $\partial \sigma_1 \sim 0$.

Suppose σ has degree < 0 . We can factor $\sigma = u^k(\sigma')$ with $\partial_J \sigma' = 0$ and degree σ' either 0 or 1. If degree σ' is 0, we have $\sigma' = \partial_J \tau$ hence $\sigma = \partial(u^k \tau)$. Now, assume that the degree of σ' is 1. If $k > 1$ we have $\sigma = u^{k-1}(u\sigma')$. Since $u\sigma'$ is a cycle

of degree > 1 , we have $\partial_J \tau = u\sigma'$ and thus $\partial_J \sigma = \partial_J(u\tau)$. Finally, we deal with the case $\sigma = u\sigma'$ where σ' has degree 1. We have $[\sigma'] = n[id]$ in $H_*^+(M)$. However, $\partial_J(pt) = id[u]$ which implies that σ is a boundary.

$H^\infty(S^1)$: By localization,

$$H^\infty(S^1) = u^{-1}H^+(S^1) = 0$$

Of course, a direct argument is also possible.

$H^-(S^1)$: This follows from the long exact sequence connecting the groups. Note that in this case $[pt]$ is a cycle since

$$\partial_J[pt] = id[u] = 0$$

and $u = 0$.

$$id : S^1 \rightarrow S^1$$

on the other hand is a boundary since $[S^1] = \partial_J[pt]u^{-1}$.

□

9.4. Relation to the Borel Construction. Let us first recall the Borel construction of equivariant homology. Let S^∞ be the unit sphere in an infinite complex Hilbert space. S^∞ is contractible and has a smooth S^1 action. We let $\mathbb{C}P^\infty$ be the quotient S^∞/S^1 . Let M be a smooth manifold with a smooth S^1 -action. The Borel model of M is defined as

$$M_{S^1} = (S^\infty \times M)/S^1$$

Let

$$H_*^B(M) = H_*(M_{S^1})$$

In this section we demonstrate a natural isomorphism

$$H_*^-(M) \cong H_*^B(M)$$

Remark. Instead of introducing infinite dimensional manifolds one may take a finite dimensional approximation S^k to S^∞ and define $H_*^B(M)$ as the limit

$$\lim_{k \rightarrow \infty} H_*((M \times S^k)/S^1)$$

As a preliminary step, we have the following lemma:

Lemma 31. *Assume that M has a free S^1 action. We have $H_*(M/S^1) \cong H_*^-(M)$ with the map induced by the projection*

$$\pi : M \rightarrow M/S^1$$

Proof. Given a chain $\sigma = \sum_{i=0}^n \sigma_i u^{-i} \in H_*(M)$, let $F_*(\sigma) = \sigma_0 \in C_*(M/S^1)$. To see that F_* is a chain map note that $F_*(J(\sigma))$ is always degenerate and hence 0 in $C_*(M/S^1)$. The rest of the argument is a standard application of the Mayer-Vietoris sequence. Let $U \subset M/S^1$ be a small ball. The restriction of π to U is equivalent to the projection $S^1 \times U \rightarrow U$. Since

$$H_*(S^1 \times U) \cong \mathbb{Z}$$

with the generator given by $p \times q$ for any $(p, q) \in U$, we have the isomorphism $H_*(S^1 \times U) \cong H_*(U)$ induced by F_* . We may cover M/S^1 by convex balls and proceed by induction to deduce the general case. \square

Corollary 1. *The projection $\pi : M \times S^\infty \rightarrow M_{S^1}$ induces an isomorphism*

$$\pi_* : H_*(M \times S^\infty) \cong H_*(M_{S^1})$$

The isomorphism $H_*(M) \cong H_*^B(M)$ now follows from the following lemma:

Lemma 32. *The equivariant projection $\pi_1 : M \times S^\infty \rightarrow M$ induces an isomorphism on homology.*

Proof. As S^∞ is contractible, the result is a consequence of the following general fact. Suppose we have an equivariant map $f : M \rightarrow N$. If

$$f_* : H_*(M) \rightarrow H_*(N)$$

is an isomorphism, then so is

$$f_* : H_*(M) \rightarrow H_*(N)$$

The proof of this fact is the following filtration argument. Let C_*^- be the chain complex of either M or N . Let $F^k C_*^- \subset C_*^-$ be the subcomplex generated by sums $\sum_{i=0}^k \sigma_i u^{-i}$. We have

$$F^k C_*^- / F^{k-1} C_*^- = C_{*-2k}$$

Therefore, since $C_a = 0$ for $a < 0$, in each degree l the filtration $F^k C_l^-$ is stable for k sufficiently large. The conclusion follows from the 5-lemma by comparing the long exact sequences for the pair $(F^k C_*^-, F^{k-1} C_*^-)$. \square

9.5. Chern Classes and Equivariant Cohomology. Assume

$$\pi : V \rightarrow M$$

is a complex vector bundle of dimension n . In this section we describe a construction of Chern classes using equivariant cohomology. In particular, transversality theory plays a major role in the construction. Observe that V has a natural S^1 -action induced by the restricting the action of \mathbb{C}^* to S^1 on each fiber. We think of M as having a trivial S^1 -action. Let

$$i : M \rightarrow V$$

be the zero section.

Theorem 21. *We have $i^* : H_+^*(V) \cong H_+^*(M) \cong H^*(M) \otimes \mathbb{Z}[v]$*

Proof. Follows from the fact that $\pi \circ i = Id$, while $i \circ \pi$ is equivariantly homotopic to the identity. \square

Recall that the Thom class U is given by $i : M \rightarrow V$. In this section, we view it as equivariant cohomology class in $H_+^*(V)$.

Definition 26. *Let $Ch(V) = i^*(U) \in H_+^*(M)$ be the total Chern class.*

Note that $Ch(V)$ decomposes as

$$[ch_n(V)] + [ch_{n-1}(V)]v + [ch_{n-2}(V)]v^2 \cdots$$

with $ch_{n-i}(V)$ of degree $2(n-i)$. Let us unravel this definition. We need to modify U to produce an equivariant class transverse to $i : M \rightarrow V$. Take τ_0 with $\partial\tau_0 \sim U - U'$ where U' is transverse to i . We have

$$U - \partial_J\tau_0 = U' - (J\tau_0)v$$

Now, pick τ_1 such that $J\tau_0 - \partial\tau_1$ is transverse to i . This time the correction term is $(J\tau_1)v^2$. Repeat until the correction term is degenerate and hence can be ignored. We have produced a chain

$$U' + a_1v + a_2v^2 + \cdots$$

such that each term is transverse to i . We can pull this back to obtain $Ch(V)$ as desired. In particular, this implies that $ch_n(V)$ is the Euler class of V .

Example. Here is the simplest version of this construction. Let us consider $\mathbb{C} \rightarrow pt$ as a line bundle over the origin. We define the homotopy

$$H : [0, 1] \rightarrow \mathbb{C}$$

with $H(t) = t \in \mathbb{C}$. The resulting section is $1 \in \mathbb{C}$ that is now transverse to 0 and its pullback is the empty set. Thus, the first Chern class is 0. According to our construction, we must examine the map

$$\sigma : S^1 \times [0, 1] \rightarrow \mathbb{C}$$

given by mapping $(e^{i\theta}, t)$ to $e^{i\theta}t$. σ does not have zero as a regular value. However, any other point in the open unit ball is a regular value of σ and has intersection 1 with the chain σ . A small perturbation of σ produces a chain σ' that has intersection 1 with the origin. Therefore, the pullback of the zero section is $1 \in H^*(pt)$. A version of this argument applies generally to a complex vector bundle $V \rightarrow M$ to conclude that $ch(V)_0 = 1$.

Lemma 33. *We have $Ch(V_1 \oplus V_2) = Ch(V_1) \cup Ch(V_2)$.*

Proof. Let U_i be the equivariant cochain in $C_+^*(V_i)$ given by the zero section

$$s_i : M \rightarrow V_i$$

Consider the vector bundle $V_1 \oplus V_2$ over $M \times M$. The cochain

$$U_1 \times U_2 \in C_+^*(V_1 \oplus V_2)$$

given by the zero section $i : M \times M \rightarrow V_1 \oplus V_2$ is the Thom class of $V_1 \oplus V_2$. Since,

$$\pi_i^* : H_+^*(M) \rightarrow H_+^*(V_i)$$

is an isomorphism, we can take $\pi_i^*(Ch(V_i)) = U_i'$ as a cocycle that also represents the Thom class U_i . In other words,

$$U_i - U_i' \sim \partial_J \tau_i$$

We claim that $U_1' \times U_2'$ represents the Thom class of $V_1 \oplus V_2$. This follows from the Leibniz rule for ∂_J . Indeed, one must check that $J(\sigma \times \tau) = J(\sigma) \times \tau$ in the special case where τ has an S^1 -action. If $\sigma : P \rightarrow V_1$ and $\tau : Q \rightarrow V_2$, the map sending $(e^{i\theta}, p, q)$ to $(e^{i\theta}, p, e^{i\theta}q)$ identifies $J(\sigma \times \tau)$ with $J(\sigma) \times \tau$. As an application of this, we have that

$$\partial_J(\tau_1 \times U_2) \sim U_1 \times U_2 - U_1' \times U_2$$

and thus $U_1 \times U_2$ is homologous to $U_1' \times U_2$. Similarly, $U_1' \times U_2$ is homologous to $U_1' \times U_2'$. The desired conclusion now follows from pulling back $Ch(V_1 \oplus V_2)$ to M via the diagonal. \square

Theorem 22. *$Ch(V)$ agrees with the usual definition of the total Chern class.*

Proof. By the splitting principle [8], it suffices to treat the case of a line bundle. \square

9.6. Steenrod Operations and Equivariant Cohomology. Let $V \rightarrow M$ be a real vector bundle. V has a natural fiberwise \mathbb{Z}_2 -action given by

$$(m, v) \mapsto (m, -v)$$

As in the complex case, the Thom class U is given by $i : M \rightarrow V$ where i is the inclusion of M as the zero section. We view U as a \mathbb{Z}_2 -equivariant cohomology class in $H_+^*(V; \mathbb{Z}_2)$. In direct analogy with our construction of Chern classes, one may define the total Stiefel-Whitney class:

Definition 27. *Let $SW(V) = i^*(U) \in H_+^*(M; \mathbb{Z}_2)$ be the total Stiefel-Whitney class.*

The verification of the product formula is identical to the case of Chern classes as discussed above.

We may also use equivariant cohomology to define Steenrod operations. Consider the action of \mathbb{Z}_2 on $M \times M$ that sends (x, y) to (y, x) . Let

$$\Delta : M \rightarrow M \times M$$

be the inclusion of M as the diagonal. Given a cohomology class $\tau : Q \rightarrow M$, note that $\tau \times \tau$ defines an equivariant class in $H_+^*(M \times M; \mathbb{Z}_2)$.

Definition 28. *The total Steenrod square class is defined as*

$$Sq(\tau) = \Delta^*(\tau \times \tau) \in H_+^*(M; \mathbb{Z}_2)$$

The top degree part of $Sq(\tau)$ is just the cup product $\tau \cup \tau$. The verification of the fundamental properties of the Steenrod operations, including the Cartan formula, is nearly identical to the case of Chern classes. In addition, a neighborhood of the diagonal in $M \times M$ may be identified with the tangent bundle of M in a manner that respects the \mathbb{Z}_2 -action. It follows that the total Stiefel-Whitney class of the tangent bundle coincides with the Steenrod square of the diagonal.

10. RELATION TO OTHER THEORIES

In this section we construct natural maps between H_* and various forms of (co)homology found in the literature. The fact that the natural maps induce isomorphisms follow from the standard Mayer-Vietoris arguments and thus we mostly restrict our attention to describing the relevant maps.

The standard construction of singular homology $H_*^{sing}(M)$ is based on mappings

$$\sigma : \Delta^n \rightarrow M$$

where Δ^n is an n -simplex. If M is smooth, one may restrict to smooth mappings ([7]). By definition, each smooth σ is a well defined chain in M . Since the definition of the boundary maps are compatible, we have a well defined map

$$H_*^{sing}(M) \rightarrow H_*(M)$$

There is a variant of singular theory that is based in mappings of cubes (see [11])

$$\sigma : [0, 1]^n \rightarrow M$$

To obtain the correct chain complex one must quotient out degenerate cubes. By definition, these are mappings σ that are independent of one of the coordinates. Note that such cubes are also degenerate from our point of view. Indeed, let's say that

$$\sigma : [0, 1]^n = [0, 1]^{n-1} \times [0, 1] \rightarrow M$$

is independent of the last coordinate. It follows that σ has small image. In addition,

$$\partial\sigma : \partial([0, 1]^{n-1}) \times [0, 1] \sqcup [0, 1]^{n-1} \times \partial[0, 1] \rightarrow M$$

is a union of a chain with small image and a trivial chain. If $H_*^{cube}(M)$ denotes cubical homology, we have defined a natural map

$$H_*^{cube}(M) \rightarrow H_*(M)$$

Let $\Omega^*(M)$ be the chain complex of smooth differential forms on M . Integration defines a map

$$\Omega_{deRham}^*(M) \rightarrow C_*(M; \mathbb{R})^*$$

To see that this map is well defined, we must check that a form ω integrates to zero on trivial chains and chains with small image. Indeed, if

$$\sigma : P \rightarrow M$$

has small image, $\sigma^*(\omega)$ vanishes since the rank of σ is less than the dimension of P at every point in the domain. If σ is trivial, let $\phi : P \rightarrow P$ be an orientation reversing diffeomorphism that commutes with σ . We have

$$\sigma^*(\omega) = \phi^* \circ \sigma^*(\omega)$$

. On the other hand, since ϕ reverses orientation,

$$\int_P \phi^* \circ \sigma^*(\omega) = - \int_P \sigma^*(\omega)$$

Therefore, $\int_P \sigma^*(\omega) = 0$.

Let us now discuss the equivariant situation. Assume that S^1 acts smoothly on M . Let ξ denote the vector field induced by the infinitesimal action. The Cartan construction gives us a deRham version of equivariant cohomology as follows. Let $\Omega_{S^1}^*(M)$ be the complex of S^1 -invariant forms. Form a new complex

$$(\Omega^*(M) \otimes \mathbb{R}[v], d_{S^1})$$

where v has degree 2 and the differential acts as

$$d_{S^1}(\omega) = d\omega + i_\xi \omega v$$

The resulting cohomology groups $H_*^{S^1, deRham}(M)$ are the Cartan model for the equivariant cohomology of M [15]. We define a chain map

$$\Omega_{deRham}^*(M) \otimes \mathbb{R}[v] \rightarrow C_*^-(M; \mathbb{R})^*$$

as follows. Given a chain $\sum_{i=0}^N \sigma_i u^{-i} \in C_k^-(M)$ with σ_i of dimension $k - 2i$ and a cochain

$$\sum_{i=0}^N \omega_i v^i \in \Omega_{deRham}^*(M) \otimes \mathbb{R}[v]$$

we have the evaluation map

$$\sum_{i=0}^N \int_{P_i} \sigma_i^*(\omega_i) \in \mathbb{R}$$

To check that this defines a chain map, we use Stoke's theorem and the fact that $\int_{S^1 \times P} \sigma^*(\omega) = \int_P \sigma^*(i_\xi \omega)$ as ω is S^1 -invariant.

11. UNIVERSAL COEFFICIENTS

The proof of the universal coefficient theorem for homology has a subtlety in our case. As defined, it is not obvious that $C_*(M)$ is free. However, we have:

Lemma 34. *For any M , $C_*(M)$ is torsion-free.*

Proof. Suppose there is $\sigma \in C_*(M)$ with $k\sigma \sim 0$ for some $k \in \mathbb{Z}^+$. As in proof of lemma 10, we decompose σ as

$$\sigma = \sigma_1 \sqcup \sigma_2 \dots$$

where each σ_i consists of the union of isomorphic connected components (up to orientation) of σ and no σ_i, σ_j share isomorphic components when $i \neq j$. Since $k\sigma \sim 0$, we have that either $k\sigma_i$ is trivial or has small image for each i . This implies that each σ_i is either trivial or has small image. We can ignore all the trivial components and assume that each σ_i has small image and thus σ has small image. By hypothesis, $k\partial\sigma \sim 0$. By repeating the argument, it follows that $\partial\sigma$ is a union of a trivial chain and a chain with small image. This implies that $\sigma \sim 0$ as it is degenerate. \square

By standard homological arguments (see [6]), $C_*(M)$ is thus a flat \mathbb{Z} -module. Suppose A is a \mathbb{Z} -module. Let $H_*(M; A)$ denote the homology of $C_*(M) \otimes_{\mathbb{Z}} A$. We may now use the standard homological arguments (see [6]) to prove the universal coefficients theorem:

Theorem 23. *For any \mathbb{Z} module A , we have the exact sequence*

$$0 \rightarrow H_*(M; \mathbb{Z}) \otimes A \rightarrow H_*(M; A) \rightarrow \text{Tor}^{\mathbb{Z}}(H_{*-1}(M; \mathbb{Z}), A) \rightarrow 0$$

12. DUALITY

Our definition of cohomology groups is geometric and does not follow the standard route of dualizing the homology chain complex. Recall that we have defined a cap product

$$H_a(M; \mathbb{Z}) \otimes H^b(M; \mathbb{Z}) \rightarrow H_{a-b}(M; \mathbb{Z})$$

by taking intersections of generic transverse representatives for our cycles. Letting $b = a$ and taking coefficients in a field \mathbb{K} , we get a map:

$$D : H^a(M; \mathbb{K}) \rightarrow \text{Hom}(H_a(M), \mathbb{K})$$

Theorem 24. *The map D is an isomorphism.*

Proof. This is a variant of the standard Mayer-Vietoris argument (see [8]). One covers M by convex balls and inducts on the number of balls in a cover. The inductive step consists of recognizing that for a open ball in \mathbb{R}^n , we have a perfect pairing between the cycle represented by a point and the cocycle represented by the entire open ball. \square

Remark. Our version of geometric duality does not assume M is compact. Now, assume that M is a compact oriented manifold of dimension n . We have an identification

$$H_*(M; \mathbb{Z}) \cong H^{n-*}(M; \mathbb{Z})$$

sending a cycle σ of dimension k to the cocycle represented by σ of codimension $n - k$. Note that an oriented chain may be viewed as a cooriented cochain using the orientation of M . From our point of view, this last isomorphism is basically a tautology since we do not define cohomology groups by dualizing homology. Therefore, the real content of duality lies in the nondegeneracy of the intersection pairing.

13. SPACES WITH A FUNCTIONAL

13.1. Basic Definitions. So far, we have worked exclusively with smooth manifolds and smooth maps. In a sense, this is not a great loss of generality since any finite CW complex has the homotopy type of a smooth manifold (possibly noncompact). However, there are certain natural geometric constructions that are not possible if one restricts to smooth spaces. For example, given a subset $S \subset M$ one may form the quotient M/S . However, even if S is a smooth submanifold, M/S does not in general inherit a smooth structure. For instance, the construction of cones and suspensions that is central to homotopy theory leads to singular spaces. One of the goals of this section is to provide a language to model these nonsmooth phenomena in the smooth category. This leads us to the notion of a space with functional and morphisms between such spaces. This formalism plays a crucial role in the construction of Floer theory in [5] and we hope it is also illuminating in finite dimensions.

Definition 29. A *space with a functional* (M, f) , is a pair where M is a smooth finite dimensional manifold without boundary and

$$f : M \rightarrow \mathbb{R}$$

is a continuous function.

The purpose of introducing f is that it provides a way of characterizing the points in M that approach $\pm\infty$. We come to the definition of homology groups of (M, f) :

Definition 30. Let P be a smooth oriented manifold with corners. A smooth map

$$\sigma : P \rightarrow M$$

is a chain if given $p_i \in P$, either p_i has a converging subsequence or

$$\lim_{i \rightarrow \infty} f(\sigma(p_i)) = -\infty$$

Intuitively, the only source of noncompactness of σ comes from points going to $-\infty$. We may now define trivial and degenerate chains as in our definition of geometric homology in section 3. We let $H_*(M, f)$ denote the resulting groups. They are graded by the dimension of the chains.

Example 1. Consider (\mathbb{R}, f) for several different choices of f . If $f = 0$, we recover the usual homology of \mathbb{R} . If $f(t) = t$ then in fact $H_*(\mathbb{R}, t) = 0$. Indeed, given a point $m \in \mathbb{R}$, we may form the homotopy

$$(-\infty, 0] \rightarrow \mathbb{R}$$

mapping $t \in (-\infty, 0]$ to $m + t$. The same homotopy shows that all 1-cycles are trivial as well. If we take $f(t) = -t^2$, we have $H_0(\mathbb{R}, -t^2) = 0$ by a similar argument. On the other hand, this time $H_1(\mathbb{R}, -t^2) = \mathbb{Z}$. It is generated by the identity map $\mathbb{R} \rightarrow \mathbb{R}$. This cycle is nontrivial since it pairs with a point to give 1.

Example 2. Given a vector bundle $V \rightarrow M$, fix a metric on V . Let $f(m, v) = -|v|^2$. One may view (V, f) as the smooth analogue of the Thom space construction. Note that any fibre defines a homology class in $H_*(V, f)$, provided V is oriented.

Example 3. Let w be a closed 1-form on M and let \widetilde{M} be a cover. Suppose that the pullback of w is an exact 1-form df . Consider (\widetilde{M}, f) . $H_*(\widetilde{M}, f)$ is the topological analogue of Novikov homology (see [9]). Indeed, in addition to compact cycles, we allow for countable sums of compact cycles as long as the images tend to $-\infty$.

Example 4. We say that (M, f) is **pointed** if each component of M contains a sequence that tends to $-\infty$. For such spaces, $H_0(M, f)$ vanishes since a point in M may be moved to $-\infty$ by a path $\gamma : (-\infty, 0] \rightarrow M$. Given any manifold M , we may form a pointed space by removing a point p_i from each component M_i and taking

$$f_i : M_i - p_i \rightarrow \mathbb{R}$$

to be a function that approaches $-\infty$ as points approach p_i . For instance, starting from S^1 we obtain $(\mathbb{R}, -t^2)$ by this procedure. Given two pointed spaces (M, f) and (N, g) we have the product $(M \times N, f + g)$. This is actually the smash product for pointed spaces. For example, $(\mathbb{R}^n, -|x|^2)$ is the n th iterated product of $(\mathbb{R}, -t^2)$. It is the pointed version of S^n .

Example 5. Our formalism makes it possible to construct cones in the smooth category. Let $S \subset M$ be a closed submanifold. Take a function

$$f : M \rightarrow [0, 1]$$

that equals 1 in a tubular neighborhood V of S and vanishes outside a larger tubular neighborhood U . Let $I = (-\infty, -1)$. Consider $(M \times I, f \cdot t)$ where t is the coordinate on \mathbb{R} . We decompose our space $M \times I$ as $(M - U) \times I$ and $U \times I$. On $(M - U) \times I$, $f \cdot t = 0$ and thus, up to homotopy, we get $M - U$. On $U \times I$, on the other hand, we claim that every cycle is nullhomotopic. Indeed, given a cycle P , first deform it to lie in $V \times I$. Now, observe that by moving the t coordinate to $-\infty$ we provide a

homotopy from our cycle to the empty set.

We also have the notion of a cochain:

Definition 31. Let Q be a smooth manifold with corners. A **cochain** $\tau : Q \rightarrow M$ is a smooth cooriented map such that given $q_i \in Q$, either q_i has a converging subsequence or $\lim f(\tau(q_i)) = \infty$.

We let $H^*(M, f)$ denote the resulting cohomology groups which are graded by codimension. Note that we have pairing

$$H^*(M, f) \otimes H_*(M, f) \rightarrow H_*(M)$$

since the intersections of a cycle and a cocycle must be compact.

The notion of cohomology of a space with functional is a generalization of usual notion of cohomology. Namely, given any smooth manifold M , let f be any exhausting function on M that is bounded below. Thus, $m_i \in M$ either has a convergent subsequence or $\lim_{i \rightarrow \infty} f(m_i) = \infty$. We associate to M the space (M, f) . In this case, we have:

Lemma 35. We have $H_*(M, f) \cong H_*(M)$ and $H^*(M, f) \cong H^*(M)$

Remark. Note that if we take $f = 0$ in the previous construction, we get cohomology with compact supports.

13.2. Morphisms.

Definition 32. Let U be open in M . A morphism $\phi : (M, f) \rightarrow (N, g)$ is a smooth map

$$\phi : U \rightarrow N$$

such that if $m_i \in U$ does not have a convergent subsequence in U , then $g(\phi(u_i)) \rightarrow -\infty$.

Lemma 36. Let ϕ be a morphism. Given a chain $\sigma : P \rightarrow M$,

$$\phi \circ \sigma|_{\sigma^{-1}(U)} : \sigma^{-1}(U) \rightarrow N$$

is a chain.

Proof. Suppose $p_i \in \sigma^{-1}(U)$ is a sequence without a convergent subsequence. We claim that $\sigma(p_i)$ does not have a convergent subsequence in U . Indeed, if some subsequence $\sigma(p_j)$ had a limit $y \in U$, then p_j must have a convergent subsequence $p_k \in P$. But this implies that $\lim p_k \in \sigma^{-1}(U)$, contradicting the assumption on p_i . Therefore, since $\sigma(p_i)$ does not have a convergent subsequence in U , $g(\phi(\sigma(p_i))) \rightarrow -\infty$ as desired. \square

Example 1. Let $S \subset M$ be a compact submanifold. Let

$$f : M - S \rightarrow \mathbb{R}$$

be a function that is bounded outside a neighborhood of S and approaches $-\infty$ as a point approaches S . We define the collapsed space to be $(M - S, f)$. The morphism

$$\phi : M \rightarrow M - S$$

defined by the identity map on $M - S$ is our smooth analogue of the collapsing map.

An important special case that is central to homotopy theory is the collapsing map

$$S^n \rightarrow S^n \vee S^n$$

that is obtained by collapsing an equatorial S^{n-1} . Here we view $S^n \vee S^n$ as a disjoint union of two open hemispheres of the original S^n .

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